The Mean and Variance Of The Product Of Two Normally-Distributed Random Variates

Gary Schurman, MBE, CFA

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We are tasked with determining the mean and variance of the product of a constant 3 times x, a normally-distributed random variate with mean 60 and standard deviation 20, and a constant 5 times y, a normally-distributed random variate with mean 25 and standard deviation 10. The correlation between x and y is 0.40.

Legend of Symbols

A	=	Multiple of x
B	=	Multiple of y
x	=	Normally-distributed random variate with mean μ_x and variance σ_x^2
y	=	Normally-distributed random variate with mean μ_y and variance σ_y^2
ρ	=	Pearson correlation coefficient between x and y
z_x	=	Normally-distributed random variate with mean zero and variance one - Used in the equation of x
z_y	=	Normally-distributed random variate with mean zero and variance one - Used in the equation of y
z	=	Normally-distributed random variate with mean zero and variance one - Used in both equations of x and y
θ	=	$\sqrt{\rho}$
ϕ	=	$\sqrt{1-\rho}$

The Equations for Ax and By

We can write the equations for x and y as follows...

$$x = \mu_x + \theta\sigma_x z + \phi\sigma_x z_x \tag{1}$$

$$y = \mu_y + \theta\sigma_y z + \phi\sigma_y z_y \tag{2}$$

The two equations above reflect the fact that x and y are correlated. We will use the remainder of this section to prove that the mean and variance of a constant times equation (1) are $A\mu_x$ and $A^2\sigma_x^2$, respectively, and the mean and variance of a constant times equation (2) are $B\mu_y$ and $B^2\sigma_y^2$, respectively, and that the correlation between x and y is ρ.

Note: If the correlation between x and y equals zero then $\theta = 0$ and $\phi = 1$ in equations (1) and (2) above.

The first moment of the distributions of Ax and Bx (see Appendix B) are...

$$\mathbb{E}\Big[Ax\Big] = A\mu_x \tag{3}$$

$$\mathbb{E}\Big[By\Big] = B\mu_y \tag{4}$$

The second moment of the distributions of Ax and By (see Appendix C) are...

$$\mathbb{E}\Big[(Ax)^2\Big] = A^2\Big[\mu_x^2 + \theta^2\sigma_x^2 + \phi^2\sigma_x^2)\Big] \tag{5}$$

$$\mathbb{E}\Big[(By)^2\Big] = B^2\Big[\mu_y^2 + \theta^2\sigma_y^2 + \phi^2\sigma_y^2)\Big] \tag{6}$$

The expected value of the product of Ax and By (see Appendix D) is...

$$\mathbb{E}\Big[AxBy\Big] = AB\Big[\mu_x\mu_y + \theta^2\sigma_x\sigma_y\Big] \tag{7}$$

The equations for the mean of Ax and the mean of By are...

$$\begin{aligned} mean_{Ax} &= \mathbb{E}\Big[Ax\Big] \\ &= A\mu_x \end{aligned} \tag{8}$$

$$\begin{aligned} mean_{By} &= \mathbb{E}\Big[By\Big] \\ &= B\mu_y \end{aligned} \tag{9}$$

The equations for the variance of Ax and the variance of By are...

$$\begin{aligned} var_{Ax} &= \mathbb{E}\Big[(Ax)^2\Big] - \Big[\mathbb{E}\Big[Ax\Big]\Big]^2 \\ &= A^2\Big[\mu_x^2 + \theta^2\sigma_x^2 + \phi^2\sigma_x^2\Big] - A^2\mu_x^2 \\ &= A^2\sigma_x^2\Big[\theta^2 + \phi^2\Big] \\ &= A^2\sigma_x^2\Big[\rho + 1 - \rho\Big] \\ &= A^2\sigma_x^2 \end{aligned} \tag{10}$$

$$\begin{aligned} var_{By} &= \mathbb{E}\Big[(By)^2\Big] - \Big[\mathbb{E}\Big[By\Big]\Big]^2 \\ &= B^2\sigma_y^2 \end{aligned} \tag{11}$$

The covariance of Ax and By is...

$$\begin{aligned} cov_{AxBy} &= \mathbb{E}\Big[AxBy\Big] - \mathbb{E}\Big[Ax\Big]\mathbb{E}\Big[By\Big] \\ &= AB\Big[\mu_x\mu_y + \theta^2\sigma_x\sigma_y\Big] - A\mu_xB\mu_y \\ &= AB\theta^2\sigma_x\sigma_y \\ &= AB\rho\sigma_x\sigma_y \end{aligned} \tag{12}$$

The correlation of Ax and By is...

$$\begin{aligned} cor_{AxBy} &= \frac{cov_{AxBy}}{\sqrt{var_{Ax}}\sqrt{var_{By}}} \\ &= \frac{AB\rho\sigma_x\sigma_y}{A\sigma_xB\sigma_y} \\ &= \rho \end{aligned} \tag{13}$$

Note: If we set $A = 1$ and $B = 1$ then the correlation between x and y is also ρ, which means that equations (1) and (2) have the desired correlation structure.

The Mean and Variance of the Product of Ax and By

The mean of the product of Ax and By is...

$$\begin{aligned} mean_{AxBy} &= \mathbb{E}\Big[AxBy\Big] \\ &= AB\Big[\mu_x\mu_y + \theta^2\sigma_x\sigma_y\Big] \\ &= AB\Big[\mu_x\mu_y + \rho\sigma_x\sigma_y\Big] \end{aligned} \tag{14}$$

The expected value of the product of Ax and By squared (see Appendix E) is...

$$\begin{aligned} \mathbb{E}\Big[(AxBy)^2\Big] = A^2B^2\Big[&\mu_x^2\mu_y^2 + \mu_x^2\theta^2\sigma_y^2 + \mu_x^2\phi^2\sigma_y^2 + 4\mu_x\mu_y\theta^2\sigma_x\sigma_y + \mu_y^2\theta^2\sigma_x^2 + 3\theta^4\sigma_x^2\sigma_y^2 + 2\theta^2\phi^2\sigma_x^2\sigma_y^2 + \\ &\mu_y^2\phi^2\sigma_x^2 + \phi^4\sigma_x^2\sigma_y^2\Big] \end{aligned} \tag{15}$$

The variance of the product of Ax and By is...

$$\begin{aligned} var_{AxBy} &= \mathbb{E}\Big[(AxBy)^2\Big] - \Big[\mathbb{E}\Big[AxBy\Big]\Big]^2 \\ &= A^2B^2\Big[\mu_x^2\mu_y^2 + \mu_x^2\theta^2\sigma_y^2 + \mu_x^2\phi^2\sigma_y^2 + 4\mu_x\mu_y\theta^2\sigma_x\sigma_y + \mu_y^2\theta^2\sigma_x^2 + 3\theta^4\sigma_x^2\sigma_y^2 + 2\theta^2\phi^2\sigma_x^2\sigma_y^2 + \\ &\quad \mu_y^2\phi^2\sigma_x^2 + \phi^4\sigma_x^2\sigma_y^2\Big] - \Big[AB\Big[\mu_x\mu_y + \theta^2\sigma_x\sigma_y\Big]\Big]^2 \\ &= A^2B^2\Big[\mu_x^2\theta^2\sigma_y^2 + \mu_x^2\phi^2\sigma_y^2 + 2\mu_x\mu_y\theta^2\sigma_x\sigma_y + \mu_y^2\theta^2\sigma_x^2 + 2\theta^4\sigma_x^2\sigma_y^2 + 2\theta^2\phi^2\sigma_x^2\sigma_y^2 + \\ &\quad \mu_y^2\phi^2\sigma_x^2 + \phi^4\sigma_x^2\sigma_y^2\Big] \\ &= A^2B^2\Big[\rho\mu_x^2\sigma_y^2 + (1-\rho)\mu_x^2\sigma_y^2 + 2\rho\mu_x\mu_y\sigma_x\sigma_y + \rho\mu_y^2\sigma_x^2 + 2\rho^2\sigma_x^2\sigma_y^2 + 2\rho(1-\rho)\sigma_x^2\sigma_y^2 + \\ &\quad (1-\rho)\mu_y^2\sigma_x^2 + (1-\rho)^2\sigma_x^2\sigma_y^2\Big] \\ &= A^2B^2\Big[\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + (1+\rho^2)\sigma_x^2\sigma_y^2 + 2\rho\mu_x\mu_y\sigma_x\sigma_y\Big] \end{aligned} \tag{16}$$

Problem Solution

We can write the equations for x and y as...

$$\begin{aligned} x &= 60 + 20\sqrt{0.40}z + 20\sqrt{1-0.40}z_x \\ y &= 25 + 10\sqrt{0.40}z + 10\sqrt{1-0.40}z_y \end{aligned} \tag{17}$$

The mean of $AxBy$ via equation (14) is...

$$\begin{aligned} mean_{AxBy} &= (3)(5)\Big[(60)(25) + (0.40)(20)(10)\Big] \\ &= 23,700 \end{aligned} \tag{18}$$

The variance of $AxBy$ via equation (16) is...

$$\begin{aligned} var_{AxBy} &= (3^2)(5^2)\Big[(60^2)(10^2) + (25^2)(20^2) + (1+0.40^2)(20^2)(10^2) + (2)(0.40)(60)(25)(20)(10)\Big] \\ &= 201,690,000 \end{aligned} \tag{19}$$

Appendix

A. Rules on expectations of random variables with mean zero and variance one:

$$\mathbb{E}\Big[z^1\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_x^1\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_y^1\Big] = 0$$

$$\mathbb{E}\Big[z^2\Big] = 1 \quad ; \quad \mathbb{E}\Big[z_x^2\Big] = 1 \quad ; \quad \mathbb{E}\Big[z_y^2\Big] = 1$$

$$\mathbb{E}\Big[z^3\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_x^3\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_y^3\Big] = 0$$

$$\mathbb{E}\Big[z^4\Big] = 3 \quad ; \quad \mathbb{E}\Big[z_x^4\Big] = 3 \quad ; \quad \mathbb{E}\Big[z_y^4\Big] = 3$$

B. The expected value of Ax is...

$$\begin{aligned}\mathbb{E}\Big[Ax\Big] &= \mathbb{E}\Big[A(\mu_x + \theta\sigma_x z + \phi\sigma_x z_x)\Big] \\ &= A\mathbb{E}\Big[\mu_x + \theta\sigma_x z + \phi\sigma_x z_x\Big] \\ &= A\mu_x \end{aligned} \tag{20}$$

One equation with three variables yields $3^1 = 3$ permutations as follows...

Perm	Eq	Value		Result
01	a1	μ_x	=	μ_x
02	a2	$\theta\sigma_x z$	=	0
03	a3	$\phi\sigma_x z_x$	=	0

Note: The same logic used above can be applied to the expected value of By

C. The expected value of Ax squared is...

$$\begin{aligned}\mathbb{E}\Big[(Ax)^2\Big] &= \mathbb{E}\Big[A(\mu_x + \theta\sigma_x z + \phi\sigma_x z_x)A(\mu_x + \theta\sigma_x z + \phi\sigma_x z_x)\Big] \\ &= A^2\mathbb{E}\Big[(\mu_x + \theta\sigma_x z + \phi\sigma_x z_x)(\mu_x + \theta\sigma_x z + \phi\sigma_x z_x)\Big] \\ &= A^2\Big[\mu_x^2 + \theta^2\sigma_x^2 + \phi^2\sigma_x^2\Big] \end{aligned} \tag{21}$$

Two equations each with three variables yields $3^2 = 9$ permutations as follows...

Perm	Eq 1	Eq 2	Value 1	Value 2		Result
01	a1	b1	μ_x	μ_x	=	μ_x^2
02	a1	b2	μ_x	$\theta\sigma_x z$	=	0
03	a1	b3	μ_x	$\phi\sigma_x z_x$	=	0
04	a1	b1	$\theta\sigma_x z$	μ_x	=	0
05	a1	b2	$\theta\sigma_x z$	$\theta\sigma_x z$	=	$\theta^2\sigma_x^2$
06	a1	b3	$\theta\sigma_x z$	$\phi\sigma_x z_x$	=	0
07	a1	b1	$\phi\sigma_x z_x$	μ_x	=	0
08	a1	b2	$\phi\sigma_x z_x$	$\theta\sigma_x z$	=	0
09	a1	b3	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	=	$\phi^2\sigma_x^2$

Note: The same logic used above can be applied to the expected value of By squared

D. The expected value of the product of Ax and By is...

$$\begin{aligned}\mathbb{E}\Big[AxBy\Big] &= \mathbb{E}\Big[A(\mu_x+\theta\sigma_x z+\phi\sigma_x z_x)B(\mu_y+\theta\sigma_y z+\phi\sigma_y z_y)\Big]\\ &= AB\mathbb{E}\Big[(\mu_x+\theta\sigma_x z+\phi\sigma_x z_x)(\mu_y+\theta\sigma_y z+\phi\sigma_y z_y)\Big]\\ &= AB\Big[\mu_x\mu_y+\theta^2\sigma_x\sigma_y\Big]\end{aligned} \tag{22}$$

Two equations each with three variables yields $3^2 = 9$ permutations as follows...

Perm	Eq 1	Eq 2	Value 1	Value 2 4		Result
01	a1	b1	μ_x	μ_y	=	$\mu_x\mu_y$
02	a1	b2	μ_x	$\theta\sigma_y z$	=	0
03	a1	b3	μ_x	$\phi\sigma_y z_y$	=	0
04	a1	b1	$\theta\sigma_x z$	μ_y	=	0
05	a1	b2	$\theta\sigma_x z$	$\theta\sigma_y z$	=	$\theta^2\sigma_x\sigma_y$
06	a1	b3	$\theta\sigma_x z$	$\phi\sigma_y z_y$	=	0
07	a1	b1	$\phi\sigma_x z_x$	μ_y	=	0
08	a1	b2	$\phi\sigma_x z_x$	$\theta\sigma_y z$	=	0
09	a1	b3	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	=	0

E. The expected value of the product of Ax and By quantity squared is...

$$\begin{aligned}\mathbb{E}\Big[(AxBy)^2\Big] &= \mathbb{E}\Big[A(\mu_x+\theta\sigma_x z+\phi\sigma_x z_x)A(\mu_x+\theta\sigma_x z+\phi\sigma_x z_x)B(\mu_y+\theta\sigma_y z+\phi\sigma_y z_y)B(\mu_y+\theta\sigma_y z+\phi\sigma_y z_y)\Big]\\ &= A^2B^2\mathbb{E}\Big[(\mu_x+\theta\sigma_x z+\phi\sigma_x z_x)(\mu_x+\theta\sigma_x z+\phi\sigma_x z_x)(\mu_y+\theta\sigma_y z+\phi\sigma_y z_y)(\mu_y+\theta\sigma_y z+\phi\sigma_y z_y)\Big]\\ &= A^2B^2\Big[\mu_x^2\mu_y^2+\mu_x^2\theta^2\sigma_y^2+\mu_x^2\phi^2\sigma_y^2+4\mu_x\mu_y\theta^2\sigma_x\sigma_y+\mu_y^2\theta^2\sigma_x^2+3\theta^4\sigma_x^2\sigma_y^2+2\theta^2\phi^2\sigma_x^2\sigma_y^2+\\ &\quad \mu_y^2\phi^2\sigma_x^2+\phi^4\sigma_x^2\sigma_y^2\Big]\end{aligned} \tag{23}$$

Four equations each with three variables yields $3^4 = 81$ permutations as follows...

Perm	Eq 1	Eq 2	Eq 3	Eq 4	Value 1	Value 2	Value 3	Value 4		Result
01	a1	b1	c1	d1	μ_x	μ_x	μ_y	μ_y	=	$\mu_x^2\mu_y^2$
02	a1	b1	c1	d2	μ_x	μ_x	μ_y	$\theta\sigma_y z$	=	0
03	a1	b1	c1	d3	μ_x	μ_x	μ_y	$\phi\sigma_y z_y$	=	0
04	a1	b1	c2	d1	μ_x	μ_x	$\theta\sigma_y z$	μ_y	=	0
05	a1	b1	c2	d2	μ_x	μ_x	$\theta\sigma_y z$	$\theta\sigma_y z$	=	$\mu_x^2\theta^2\sigma_y^2$
06	a1	b1	c2	d3	μ_x	μ_x	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
07	a1	b1	c3	d1	μ_x	μ_x	$\phi\sigma_y z_y$	μ_y	=	0
08	a1	b1	c3	d2	μ_x	μ_x	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
09	a1	b1	c3	d3	μ_x	μ_x	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	$\mu_x^2\phi^2\sigma_y^2$
10	a1	b2	c1	d1	μ_x	$\theta\sigma_x z$	μ_y	μ_y	=	0
11	a1	b2	c1	d2	μ_x	$\theta\sigma_x z$	μ_y	$\theta\sigma_y z$	=	$\mu_x\mu_y\theta^2\sigma_x\sigma_y$
12	a1	b2	c1	d3	μ_x	$\theta\sigma_x z$	μ_y	$\phi\sigma_y z_y$	=	0
13	a1	b2	c2	d1	μ_x	$\theta\sigma_x z$	$\theta\sigma_y z$	μ_y	=	$\mu_x\mu_y\theta^2\sigma_x\sigma_y$
14	a1	b2	c2	d2	μ_x	$\theta\sigma_x z$	$\theta\sigma_y z$	$\theta\sigma_y z$	=	0
15	a1	b2	c2	d3	μ_x	$\theta\sigma_x z$	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
16	a1	b2	c3	d1	μ_x	$\theta\sigma_x z$	$\phi\sigma_y z_y$	μ_y	=	0
17	a1	b2	c3	d2	μ_x	$\theta\sigma_x z$	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
18	a1	b2	c3	d3	μ_x	$\theta\sigma_x z$	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	0
19	a1	b3	c1	d1	μ_x	$\phi\sigma_x z_x$	μ_y	μ_y	=	0
20	a1	b3	c1	d2	μ_x	$\phi\sigma_x z_x$	μ_y	$\theta\sigma_y z$	=	0

21	a1	b3	c1	d3	μ_x	$\phi\sigma_x z_x$	μ_y	$\phi\sigma_y z_y$	=	0
22	a1	b3	c2	d1	μ_x	$\phi\sigma_x z_x$	$\theta\sigma_y z$	μ_y	=	0
23	a1	b3	c2	d2	μ_x	$\phi\sigma_x z_x$	$\theta\sigma_y z$	$\theta\sigma_y z$	=	0
24	a1	b3	c2	d3	μ_x	$\phi\sigma_x z_x$	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
25	a1	b3	c3	d1	μ_x	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	μ_y	=	0
26	a1	b3	c3	d2	μ_x	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
27	a1	b3	c3	d3	μ_x	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	0
28	a2	b1	c1	d1	$\theta\sigma_x z$	μ_x	μ_y	μ_y	=	0
29	a2	b1	c1	d2	$\theta\sigma_x z$	μ_x	μ_y	$\theta\sigma_y z$	=	$\mu_x\mu_y\theta^2\sigma_x\sigma_y$
30	a2	b1	c1	d3	$\theta\sigma_x z$	μ_x	μ_y	$\phi\sigma_y z_y$	=	0
31	a2	b1	c2	d1	$\theta\sigma_x z$	μ_x	$\theta\sigma_y z$	μ_y	=	$\mu_x\mu_y\theta^2\sigma_x\sigma_y$
32	a2	b1	c2	d2	$\theta\sigma_x z$	μ_x	$\theta\sigma_y z$	$\theta\sigma_y z$	=	0
33	a2	b1	c2	d3	$\theta\sigma_x z$	μ_x	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
34	a2	b1	c3	d1	$\theta\sigma_x z$	μ_x	$\phi\sigma_y z_y$	μ_y	=	0
35	a2	b1	c3	d2	$\theta\sigma_x z$	μ_x	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
36	a2	b1	c3	d3	$\theta\sigma_x z$	μ_x	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	0
37	a2	b2	c1	d1	$\theta\sigma_x z$	$\theta\sigma_x z$	μ_y	μ_y	=	$\mu_y^2\theta^2\sigma_x^2$
38	a2	b2	c1	d2	$\theta\sigma_x z$	$\theta\sigma_x z$	μ_y	$\theta\sigma_y z$	=	0
39	a2	b2	c1	d3	$\theta\sigma_x z$	$\theta\sigma_x z$	μ_y	$\phi\sigma_y z_y$	=	0
40	a2	b2	c2	d1	$\theta\sigma_x z$	$\theta\sigma_x z$	$\theta\sigma_y z$	μ_y	=	0
41	a2	b2	c2	d2	$\theta\sigma_x z$	$\theta\sigma_x z$	$\theta\sigma_y z$	$\theta\sigma_y z$	=	$3\theta^4\sigma_x^2\sigma_y^2$
42	a2	b2	c2	d3	$\theta\sigma_x z$	$\theta\sigma_x z$	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
43	a2	b2	c3	d1	$\theta\sigma_x z$	$\theta\sigma_x z$	$\phi\sigma_y z_y$	μ_y	=	0
44	a2	b2	c3	d2	$\theta\sigma_x z$	$\theta\sigma_x z$	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
45	a2	b2	c3	d3	$\theta\sigma_x z$	$\theta\sigma_x z$	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	$\theta^2\phi^2\sigma_x^2\sigma_y^2$
46	a2	b3	c1	d1	$\theta\sigma_x z$	$\phi\sigma_x z_x$	μ_y	μ_y	=	0
47	a2	b3	c1	d2	$\theta\sigma_x z$	$\phi\sigma_x z_x$	μ_y	$\theta\sigma_y z$	=	0
48	a2	b3	c1	d3	$\theta\sigma_x z$	$\phi\sigma_x z_x$	μ_y	$\phi\sigma_y z_y$	=	0
49	a2	b3	c2	d1	$\theta\sigma_x z$	$\phi\sigma_x z_x$	$\theta\sigma_y z$	μ_y	=	0
50	a2	b3	c2	d2	$\theta\sigma_x z$	$\phi\sigma_x z_x$	$\theta\sigma_y z$	$\theta\sigma_y z$	=	0
51	a2	b3	c2	d3	$\theta\sigma_x z$	$\phi\sigma_x z_x$	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
52	a2	b3	c3	d1	$\theta\sigma_x z$	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	μ_y	=	0
53	a2	b3	c3	d2	$\theta\sigma_x z$	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
54	a2	b3	c3	d3	$\theta\sigma_x z$	$\phi\sigma_x z_x$	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	0
55	a3	b1	c1	d1	$\phi\sigma_x z_x$	μ_x	μ_y	μ_y	=	0
56	a3	b1	c1	d2	$\phi\sigma_x z_x$	μ_x	μ_y	$\theta\sigma_y z$	=	0
57	a3	b1	c1	d3	$\phi\sigma_x z_x$	μ_x	μ_y	$\phi\sigma_y z_y$	=	0
58	a3	b1	c2	d1	$\phi\sigma_x z_x$	μ_x	$\theta\sigma_y z$	μ_y	=	0
59	a3	b1	c2	d2	$\phi\sigma_x z_x$	μ_x	$\theta\sigma_y z$	$\theta\sigma_y z$	=	0
60	a3	b1	c2	d3	$\phi\sigma_x z_x$	μ_x	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
61	a3	b1	c3	d1	$\phi\sigma_x z_x$	μ_x	$\phi\sigma_y z_y$	μ_y	=	0
62	a3	b1	c3	d2	$\phi\sigma_x z_x$	μ_x	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
63	a3	b1	c3	d3	$\phi\sigma_x z_x$	μ_x	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	0
64	a3	b2	c1	d1	$\phi\sigma_x z_x$	$\theta\sigma_x z$	μ_y	μ_y	=	0
65	a3	b2	c1	d2	$\phi\sigma_x z_x$	$\theta\sigma_x z$	μ_y	$\theta\sigma_y z$	=	0
66	a3	b2	c1	d3	$\phi\sigma_x z_x$	$\theta\sigma_x z$	μ_y	$\phi\sigma_y z_y$	=	0
67	a3	b2	c2	d1	$\phi\sigma_x z_x$	$\theta\sigma_x z$	$\theta\sigma_y z$	μ_y	=	0
68	a3	b2	c2	d2	$\phi\sigma_x z_x$	$\theta\sigma_x z$	$\theta\sigma_y z$	$\theta\sigma_y z$	=	0
69	a3	b2	c2	d3	$\phi\sigma_x z_x$	$\theta\sigma_x z$	$\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
70	a3	b2	c3	d1	$\phi\sigma_x z_x$	$\theta\sigma_x z$	$\phi\sigma_y z_y$	μ_y	=	0
71	a3	b2	c3	d2	$\phi\sigma_x z_x$	$\theta\sigma_x z$	$\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
72	a3	b2	c3	d3	$\phi\sigma_x z_x$	$\theta\sigma_x z$	$\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	0
73	a3	b3	c1	d1	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	μ_y	μ_y	=	$\mu_y^2\phi^2\sigma_x^2$
74	a3	b3	c1	d2	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	μ_y	$\theta\sigma_y z$	=	0
75	a3	b3	c1	d3	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	μ_y	$\phi\sigma_y z_y$	=	0
76	a3	b3	c2	d1	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	$\theta\sigma_y z$	μ_y	=	0

77	a3	b3	c2	d2	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	$\theta\sigma_y z$	$\theta\sigma_y z$	=	$\theta^2\phi^2\sigma_x^2\sigma_y^2$
78	a3	b3	c2	d3	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	$B\theta\sigma_y z$	$\phi\sigma_y z_y$	=	0
79	a3	b3	c3	d1	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	$B\phi\sigma_y z_y$	μ_y	=	0
80	a3	b3	c3	d2	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	$B\phi\sigma_y z_y$	$\theta\sigma_y z$	=	0
81	a3	b3	c3	d3	$\phi\sigma_x z_x$	$\phi\sigma_x z_x$	$B\phi\sigma_y z_y$	$\phi\sigma_y z_y$	=	$\phi^2\sigma_x^2\sigma_y^2$